On orthogonal polynomials related to arithmetic and harmonic sequences

Adhemar Bultheel and Andreas Lasarow

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KU Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

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In this paper we study special systems of orthogonal polynomials on the unit circle. More precisely, with a view to the recurrence relations fulfilled by these orthogonal systems, we analyze a link of non-negative arithmetic to harmonic sequences as a main subject. Here, arithmetic sequences appear as coefficients of orthogonal polynomials and harmonic sequences as corresponding Szegő parameters.

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MSC 2010 : Primary : 42C05; Secondary : 30C15.

Research Article

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1 Introduction

Throughout the paper, let n be a positive integer. Suppose that p is a (complex-valued) polynomial of degree n which admits the representation

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

with non-negative real coefficients. In this paper, especially, we are interested in the case that the difference of consecutive coefficients is positive and constant, i.e.

$$a_k - a_{k-1} = d, \qquad k = 1, \ldots, n,$$

for some (arbitrary, but fixed) positive real number d. We will denote by $\mathcal{P}_{n;\mathsf{ar}}$ the set of all polynomials of this type. The coefficients are related to arithmetic sequences (hence the "ar" in the notation) and, since $d > 0$, we have the monotonicity

$$a_n > a_{n-1} > \cdots > a_1 > a_0 \geq 0.$$

Thereby, the considerations below can be seen as a continuation of those in [2] and [5] on special polynomials appearing in orthogonal systems on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ of the complex plane $\mathbb{C}$.

Adhemar Bultheel, Departement Computerwetenschappen, KU Leuven, Belgium; e-mail: adhemar.bultheel@cs.kuleuven.be

Andreas Lasarow, Fakultät Informatik, Mathematik und Naturwissenschaften, HTWK Leipzig, Germany; e-mail: andreas.lasarow@htwk-leipzig.de

Suppose that μ is a measure belonging to $\mathcal{M}$, where $\mathcal{M}$ stands for the set of all finite measures defined on the σ-algebra $\mathfrak{B}_{\mathbb{T}}$ of all Borel subsets of $\mathbb{T}$. We will call a (finite or infinite) sequence $(\phi_k)_{k=0}^{\tau}$, where each ϕ_k is a polynomial of degree not greater than k, an *orthonormal polynomial system for* μ when

$$\int_{\mathbb{T}} \phi_s(z)\overline{\phi_t(z)}\,\mu(\mathrm{d}z) \;=\; \delta_{st},$$

where $\delta_{st} := 1$ if $s = t$ and $\delta_{st} := 0$ if $s \neq t$ (for each choice of indices). Here and henceforth, τ is a non-negative integer or $\tau = \infty$ (arbitrarily chosen, but fixed).

If there exists an orthonormal polynomial system for some $\mu \in \mathcal{M}$, then we find a special one which is uniquely determined by the extra condition that the leading coefficient of each ϕ_k is a positive real number. We will call this $(\phi_k)_{k=0}^{\tau}$ the (up to τ) *normalized* orthonormal polynomial system for μ.

As an aside, we remark that there are explicit descriptions of normalized orthonormal polynomial systems by certain determinant formulas or (equivalent) by using entries of the inverse of Toeplitz matrices given by the Fourier coefficients of the measure μ (see, e.g., [7], [6], [3], and [4, Section 3.6]). Since we are more interested in the recurrence relations for such systems, we omit here the details.

Orthonormal polynomial systems for some $\mu \in \mathcal{M}$ fulfill specific recurrence relations, where the element ϕ_n can be calculated based on ϕ_{n-1} and vice versa. By the degree of freedom of the choice of such orthonormal system one can successively choose the elements so that the related recursions only depend in each step on some parameter from the unit disk $\mathbb{D} := \{w \in \mathbb{C} :\ |w| < 1\}$ (cf. [7], [6, Sections 1.5 and 1.7], but below we follow more the approach of [3], [4, Section 3.6]).

Concerning the recurrence relations in question, the following term will be taken center stage. We will call $(\varphi_k)_{k=0}^{\tau}$ a sequence of *Szegő polynomials* when φ_0 is a constant function on $\mathbb{C}$ with a value $p_0 \in \mathbb{C} \setminus \{0\}$ and the other polynomials in the sequence with higher index are (in each n-th step) connected via

$$\varphi_n(x) \;=\; \frac{1}{\sqrt{1-|e_n|^2}}\Big(x\varphi_{n-1}(x) + e_n\widetilde{\varphi}_{n-1}^{[n-1]}(x)\Big) \tag{1}$$

with some *Szegő parameter* $e_n \in \mathbb{D}$ (cf. [3] and [4, Definition 3.6.7]). Herein and furthermore (with some non-negative integer m), the notation $\widetilde{p}^{[m]}$ stands for the polynomial which is uniquely determined via

$$\widetilde{p}^{[m]}(x) \;=\; x^m\overline{p\Big(\tfrac{1}{\overline{x}}\Big)}, \quad x \in \mathbb{C}\setminus\{0\}, \tag{2}$$

for a polynomial p of degree not greater than m. In addition, a sequence $(\varphi_k)_{k=0}^{\tau}$ of Szegő polynomials is called *canonical* when φ_0 is a constant function on $\mathbb{C}$ with some positive real number p_0 as value (cf. [4, Definition 3.6.4]).

In Section 2 we will give some information about sequences of Szegő polynomials which are known, but we recall them because that is useful for our main results. Section 3 forms the body of this paper, where we study the set $\mathcal{P}_{n;\mathsf{ar}}$ with a view to the recurrence relations fulfilled by sequences of Szegő polynomials. This approach is similar to the investigations in [2] and [5] between properties of coefficients of Szegő polynomials and corresponding properties of Szegő parameters. However, as a main result in Section 3, it will be revealed that an arithmetic sequence of coefficients is related to a harmonic sequence of Szegő parameters.

Finally, we remark that the restriction of the case that the polynomials have non-negative real coefficients is not essential. However, the calculations are somewhat more labor-intensive for the more general situation. We will give the details later.

2 Hints on sequences of Szegő polynomials

We give in this section some notes on a sequence $(\varphi_k)_{k=0}^{\tau}$ of Szegő polynomials which are useful for our main results. Even if we mostly fix the statements below only as a remark, there is usually more than one line needed to prove them precisely (depending on the knowledge).

Since the k-th element φ_k of a sequence of Szegő polynomials is of degree not greater than k (in fact, of exact degree k; cf. [2, Lemma 2.4]), there are some coefficients $a_{k;0}, a_{k;1}, \ldots, a_{k;k} \in \mathbb{C}$ so that

$$\varphi_k(x) = \sum_{j=0}^{k} a_{k;j} x^j. \tag{3}$$

Especially, we are interested in this paper in the case that all coefficients of the corresponding polynomials are non-negative real.

Remark 2.1. Suppose that φ_{n-1} is a polynomial of degree $n-1$ with non-negative real coefficients. If φ_n is the polynomial given by (1) with some real number e_n, where $0 \leq e_n < 1$, then

$$\varphi_n(x) = \sum_{j=0}^{n} \frac{a_{n-1;j-1} + e_n a_{n-1;n-j-1}}{\sqrt{1-e_n^2}} x^j$$

using the notation of the coefficients as in (3) for φ_{n-1} and setting $a_{n-1;-1} := 0$. In particular, we can see, that all coefficients of the polynomial φ_n are non-negative real in this situation as well.

The following example can be seen as the initial point of the considerations in this paper. Note that the structure of the parameters which appear in this example are closely related to that in [6, Examples 1.6.3 and 1.6.4] concerning (infinite) sequences of orthogonal polynomials on the unit circle.

Example 2.1. Let φ_0 be the constant function with value 1 and let

$$e_\ell := \frac{1}{\ell+1}, \qquad \ell = 1,2,3,4.$$

Then the (canonical) sequence $(\varphi_k)_{k=0}^4$ of Szegő polynomials with φ_0 and the sequence of Szegő parameters $(e_\ell)_{\ell=1}^4$ is given by

$$\begin{aligned}
\varphi_0(x) &= 1,\\
\varphi_1(x) &= \frac{1}{\sqrt{3}}(2x+1),\\
\varphi_2(x) &= \frac{1}{\sqrt{6}}(3x^2+2x+1),\\
\varphi_3(x) &= \frac{1}{\sqrt{10}}(4x^3+3x^2+2x+1),\\
\varphi_4(x) &= \frac{1}{\sqrt{15}}(5x^4+4x^3+3x^2+2x+1).
\end{aligned}$$

Example 2.1 suggests a link between a harmonic sequence of Szegő parameters and corresponding arithmetic sequences of coefficients of Szegő polynomials. This link will be studied in Section 3 in somewhat more detail.

Note that (1) is equivalent to

$$x\varphi_{n-1}(x) = \frac{1}{\sqrt{1-|e_n|^2}}\Big(\varphi_n(x) - e_n\widetilde{\varphi}_n^{[n]}(x)\Big) \tag{4}$$

(cf. [2, Lemma 2.4]). Thus, in addition to Remark 2.1, it follows from (1) that the polynomial φ_{n-1} can also be expressed in terms of the coefficients of the polynomial φ_n. Using the notation of the coefficients as in (3) for φ_n, we get the following.

Remark 2.2. Suppose that φ_{n-1} is a polynomial of degree $n-1$ and that φ_n is the polynomial given by (1) with non-negative real coefficients and with some $e_n \in \mathbb{D}$. Then we get $a_{n;n} > a_{n;0} \geq 0$ and $0 \leq e_n < 1$, where

$$\varphi_{n-1}(x) = \sum_{j=0}^{n-1} \frac{a_{n;n}a_{n;j+1} - a_{n;0}a_{n;n-j-1}}{\sqrt{a_{n;n}^2 - a_{n;0}^2}} x^j \qquad \text{and} \qquad e_n = \frac{a_{n;0}}{a_{n;n}}$$

(cf. [2, Lemma 2.6]). In particular, if we know the coefficients of the polynomial φ_n, then the parameter e_n and the polynomial φ_{n-1} are uniquely determined.

The next example illustrates that the case $n = 1$ concerning (1) and sequences of Szegő polynomials is unspectacular (but an exception).

Example 2.2. Suppose that p is a polynomial admitting $p(x) = a_1 x + a_0$ with some non-negative real numbers a_1 and a_0. With a view to Remark 2.2 one can see that there is a $(\varphi_k)_{k=0}^1$ of Szegő polynomials with $\varphi_1 = p$ if and only if $a_1 > a_0$, where the Szegő polynomials with $\varphi_1 = p$ is then uniquely determined and canonical (since φ_0 is the constant function with value $\sqrt{a_1^2 - a_0^2}$).

With the following example we will emphasize that, under the exclusive terms of Remark 2.2, it is possible that a coefficient of the polynomial φ_{n-1} is negative real.

Example 2.3. If φ_2 and φ_3 are the polynomials given by

$$\varphi_2(x) \; = \; 8x^2 - x + 7 \qquad \text{and} \qquad \varphi_3(x) \; = \; 10x^3 + 4x^2 + 8x + 6,$$

then (1) is fulfilled with $n = 3$ and $e_n = \frac{3}{5}$.

In view of the definition of sequences of Szegő polynomials, a finite sequence of this type can be extended to an infinite one (by choosing the missing Szegő parameters arbitrary, but belonging to $\mathbb{D}$). The following note points out a simple extension regarding (infinite) orthonormal polynomial systems by fixing the measure μ.

Remark 2.3. Suppose that $(\varphi_k)_{k=0}^n$ is a sequence of Szegő polynomials. Then $(\varphi_k)_{k=0}^n$ is an orthonormal polynomial system for μ given by

$$\mu(B) := \frac{1}{2\pi} \int\limits_B \frac{1}{|\varphi_n(z)|^2} \, \lambda(\mathrm{d}z), \quad B \in \mathfrak{B}_{\mathbb{T}}, \tag{5}$$

where λ stands for the linear Lebesgue-Borel measure on $\mathbb{T}$. In fact (cf. [6, Theorems 1.7.5 and 1.7.8] or [2, Proposition 2.5 and Remark 5.2]), if we choose

$$\varphi_{n+\ell}(x) := x^\ell \varphi_n(x), \quad \ell = 1, 2, \dots ,$$

then $(\varphi_k)_{k=0}^\infty$ is a sequence of Szegő polynomials with parameter $e_{n+\ell} = 0$ for all integers $\ell \geq 1$, where $(\varphi_k)_{k=0}^\infty$ is an orthonormal polynomial system for μ.

If $(\varphi_k)_{k=0}^n$ is a sequence of Szegő polynomials, then Remark 2.3 clarifies particularly that there is a $\mu \in \mathcal{M}$ so that $(\varphi_k)_{k=0}^n$ is an orthonormal polynomial system for μ. In addition to that (cf. [6, Theorem 1.7.11] or [4, Theorem 3.6.2]), if we consider an infinite sequence $(\varphi_k)_{k=0}^\infty$ of Szegő polynomials, then there is exactly one measure $\mu \in \mathcal{M}$ so that $(\varphi_k)_{k=0}^\infty$ is an orthonormal polynomial system for μ.

Note that, conversely, if we have an orthonormal polynomial system $(\phi_k)_{k=0}^\tau$ for $\mu \in \mathcal{M}$, then in the set of all such systems are included sequences of Szegő polynomials and we find a special one which is canonical (cf. [6, Chapter 1] or [4, Section 3.6]).

Finally, we recall the following manipulation by multiplication of sequences of Szegő polynomials (cf. [2, Remark 2.10]).

Remark 2.4. Suppose that $(\varphi_k)_{k=0}^{\tau}$ is a sequence of Szegő polynomials with related sequence $(e_\ell)_{\ell=1}^{\tau}$ of Szegő parameters and let a be a positive real number. Then $(a\varphi_k)_{k=0}^{\tau}$ is a sequence of Szegő polynomials with sequence $(e_\ell)_{\ell=1}^{\tau}$ of Szegő parameters. Furthermore, if $(\varphi_k)_{k=0}^{\tau}$ is an orthonormal polynomial system for the measure μ, then $(a\varphi_k)_{k=0}^{\tau}$ is an orthonormal polynomial system for $\frac{1}{a^2}\mu$.

3 On Szegő polynomials belonging to $\mathcal{P}_{n;\mathsf{ar}}$

Now, with a view to the set $\mathcal{P}_{n;\mathsf{ar}}$, we study special sequences of Szegő polynomials. Thereby, the following result is the lynchpin.

Lemma 3.1. *Suppose that φ_{n-1} is a polynomial of degree $n-1$ and that φ_n is the polynomial given by* (1) *with some $e_n \in \mathbb{D}$ and $n \geq 2$. If $\varphi_n \in \mathcal{P}_{n;\mathsf{ar}}$, then $\varphi_{n-1} \in \mathcal{P}_{n-1;\mathsf{ar}}$ and, using the notation of the coefficients as in* (3) *for φ_n, then*

$$\varphi_{n-1}(x) = \sum_{j=0}^{n-1} \frac{(j+1)d(nd+2a_{n;0})}{\sqrt{a_{n;n}^2 - a_{n;0}^2}} x^j, \tag{6}$$

$$e_n = \frac{a_{n;0}}{nd+a_{n;0}} \tag{7}$$

with $d = a_{n;1} - a_{n;0} > 0$.

Proof. Let $\varphi_n \in \mathcal{P}_{n;\mathsf{ar}}$. With a view to (3) for φ_n, we get $a_{n;n} > a_{n;0} \geq 0$,

$$\varphi_{n-1}(x) = \sum_{j=0}^{n-1} \frac{a_{n;n}a_{n;j+1} - a_{n;0}a_{n;n-j-1}}{\sqrt{a_{n;n}^2 - a_{n;0}^2}} x^j, \tag{8}$$

and

$$e_n = \frac{a_{n;0}}{a_{n;n}} \tag{9}$$

(cf. Remark 2.2). Furthermore, since $\varphi_n \in \mathcal{P}_{n;\mathsf{ar}}$, there is a $d > 0$ so that

$$a_{n;k} - a_{n;k-1} = d, \qquad k = 1, \ldots, n.$$

Thus, it follows that

$$e_n = \frac{a_{n;0}}{a_{n;n}} = \frac{a_{n;0}}{nd+a_{n;0}},$$

i.e. (7), and

$$\begin{aligned} a_{n;n}a_{n;j+1} - a_{n;0}a_{n;n-j-1} &= (nd+a_{n;0})\big((j+1)d+a_{n;0}\big) - a_{n;0}\big((n-j-1)d+a_{n;0}\big) \\ &= n(j+1)d^2 + nda_{n;0} + (j+1)da_{n;0} - (n-j-1)da_{n;0} \\ &= (j+1)d(nd+2a_{n;0}) \end{aligned}$$

for $j = 0, 1, \ldots, n-1$, i.e. we get (6). In particular, we see $\varphi_{n-1} \in \mathcal{P}_{n-1;\mathsf{ar}}$. □

Theorem 3.1. *Suppose that $p \in \mathcal{P}_{n;\mathsf{ar}}$ for some $n \geq 2$. Then there is a uniquely determined sequence $(\varphi_k)_{k=0}^n$ of Szegő polynomials, where $\varphi_n = p$. This sequence $(\varphi_k)_{k=0}^n$ of Szegő polynomials is canonical, where the associated Szegő parameter e_n is given via (7) and (3). Furthermore, the polynomial φ_ℓ belongs to $\mathcal{P}_{\ell;\mathsf{ar}}$ and the associated Szegő parameter e_ℓ is given by*

$$e_\ell = \frac{1}{\ell+1}, \quad \ell = 1, \ldots, n-1.$$

Proof. We set $\varphi_n = p$ and use the notation (3) with k replaced by n. Because $p \in \mathcal{P}_{n;\mathsf{ar}}$, we have $a_{n;n} > a_{n;0} \geq 0$ so that the parameter e_n and the polynomial φ_{n-1} according to (9) and (8), respectively, are well-defined. Since the relation (4) is equivalent to (1), from (8) and (9) we get (1). Furthermore, Lemma 3.1 yields $\varphi_{n-1} \in \mathcal{P}_{n-1;\mathsf{ar}}$ and the representation (7) and (6) for e_n and φ_{n-1}, respectively. In particular, if $n-1 \geq 2$, we can proceed with the approach and get $\varphi_{n-2} \in \mathcal{P}_{n-2;\mathsf{ar}}$, where Remark 2.2 and (6) imply

$$e_{n-1} = \frac{a_{n-1;0}}{a_{n-1;n-1}} = \frac{\frac{d(nd+2a_{n;0})}{\sqrt{a_{n;n}^2-a_{n;0}^2}}}{\frac{nd(nd+2a_{n;0})}{\sqrt{a_{n;n}^2-a_{n;0}^2}}} = \frac{d(nd+2a_{n;0})}{nd(nd+2a_{n;0})} = \frac{1}{n}.$$

For $\ell = 1, \ldots, n-1$, by the principle of induction, we get $\varphi_\ell \in \mathcal{P}_{\ell;\mathsf{ar}}$ and $e_\ell = \frac{1}{\ell+1}$. Finally, with a view to Example 2.2, one can see that the construction used above leads to a uniquely determined sequence $(\varphi_k)_{k=0}^n$ of Szegő polynomials, where $\varphi_n = p$, and that this is canonical. □

Corollary 3.1. *If $p \in \mathcal{P}_{n;\mathsf{ar}}$ with $n \geq 1$, then all zeros of p belong to $\mathbb{D}$.*

Proof. For $n = 1$, the statement follows immediately from the definition of $\mathcal{P}_{1;\mathsf{ar}}$. If $n \geq 2$, then Theorem 3.1 implies that there is a sequence $(\varphi_k)_{k=0}^n$ of Szegő polynomials with $\varphi_n = p$. Thus, in this case, the statement follows from a general result on sequences of Szegő polynomials (see, e.g., [2, Proposition 2.5 (a)]). □

As an aside, we note that the statement of Corollary 3.1 follows from a classical theorem due to Eneström–Kakeya as well (see, e.g., [1, Theorem A]).

The result, that is revealed by Theorem 3.1, comprises a very special structure of Szegő polynomials. This will be emphasized by the following.

Proposition 3.1. *Suppose that $p \in \mathcal{P}_{n;\mathsf{ar}}$ for some $n \geq 2$ and let $(\varphi_k)_{k=0}^n$ be the uniquely determined sequence of Szegő polynomials, where $\varphi_n = p$.*

(a) *The sequence $(\varphi_k)_{k=0}^n$ is the (up to n) normalized orthonormal polynomial system corresponding to the measure μ given by* (5).

(b) *Let d_m be the difference of consecutive coefficients of φ_m for $m = 1, 2, \ldots, n$ and let $d_0 := \varphi_0(0)$. Then $d_0 > d_1 > \cdots > d_{n-1} \geq d_n$, where $d_\ell = \varphi_\ell(0)$ and $d_{\ell-1} = d_\ell\sqrt{1+\frac{2}{\ell}}$ for $\ell = 1, \ldots, n-1$ and where $d_{n-1} = d_n\sqrt{1+\frac{2\varphi_n(0)}{nd_n}}$. Furthermore, $d_m = \frac{1}{m}\big(\widetilde{\varphi}_m^{[m]}(0) - \varphi_m(0)\big)$ for $m = 1, 2, \ldots, n$.*

(c) *Denoting the coefficients as in (3), then*

$$a_{k;j} = \frac{1}{j+1}\sqrt{\frac{n+1}{k^2+3k+2}\Big(\big(\widetilde{p}^{[n]}(0)\big)^2 - \big(p(0)\big)^2\Big)}, \quad j = 0, 1, \ldots, k,$$

for all indices $k = 0, 1, \ldots, n-1$.

(d) *The following statements are equivalent:*
  - (i) $p(0) = 0$.
  - (ii) $e_n = 0$.
  - (iii) $d_n = d_{n-1}$.

(e) *The following statements are equivalent:*
  - (iv) $p(0) = d_n$.
  - (v) $e_n = \frac{1}{n+1}$.
  - (vi) $d_n = d_{n-1}\sqrt{\frac{n}{n+2}}$.

Proof. As is well-known (cf. Remark 2.3), the sequence $(\varphi_k)_{k=0}^n$ is an orthonormal polynomial system for the measure μ given by (5). Since $\varphi_n = p$ particularly implies $\varphi_n \in \mathcal{P}_{n;\mathsf{ar}}$ and since Theorem 3.1 yields $\varphi_\ell \in \mathcal{P}_{\ell;\mathsf{ar}}$ for $\ell = 1, \ldots, n-1$ as well as that $\varphi_0(0)$ is a positive real number, the leading coefficient of φ_k is a positive real number for each $k = 0, 1, \ldots, n$. Thus, the sequence $(\varphi_k)_{k=0}^n$ is the (up to n) normalized orthonormal polynomial system for μ. Hence, (a) is proven.

Because $\varphi_m \in \mathcal{P}_{m;\mathsf{ar}}$, the number d_m is well-defined according to (b) and $\widetilde{\varphi}_m^{[m]}(0)$ is the leading coefficient of φ_m for $m = 1, 2, \ldots, n$. Consequently, based on (3), for $m = 1, 2, \ldots, n$ we have

$$\widetilde{\varphi}_m^{[m]}(0) = a_{m;m} = md_m + a_{m;0} = md_m + \varphi_m(0),$$

i.e. $d_m = \frac{1}{m}\big(\widetilde{\varphi}_m^{[m]}(0) - \varphi_m(0)\big)$. Moreover, by Theorem 3.1 and Lemma 3.1 follows the representation (6) for φ_{n-1}, where $d = d_n$. This implies

$$\begin{aligned} d_{n-1} &= \frac{d_n(nd_n + 2a_{n;0})}{\sqrt{a_{n;n}^2 - a_{n;0}^2}} = \frac{d_n(nd_n + 2a_{n;0})}{\sqrt{(nd_n + a_{n;0})^2 - a_{n;0}^2}} = \frac{d_n(nd_n + 2a_{n;0})}{\sqrt{(nd_n)^2 + 2nd_na_{n;0}}} \\ &= \frac{nd_n^2(1+\frac{2a_{n;0}}{nd_n})}{nd_n\sqrt{1+\frac{2a_{n;0}}{nd_n}}} = d_n\sqrt{1+\frac{2a_{n;0}}{nd_n}} = d_n\sqrt{1+\frac{2\varphi_n(0)}{nd_n}}. \end{aligned}$$

In particular, taking $\varphi_n(0) \geq 0$ and $d_n > 0$ into account, we see $d_{n-1} \geq d_n$. Let $\ell = 1, \ldots, n-1$. Theorem 3.1 yields $e_\ell = \frac{1}{\ell+1}$ on the one hand and on the other hand $e_\ell = \frac{a_{\ell;0}}{\ell d_\ell + a_{\ell;0}}$ by Lemma 3.1 and Remark 2.2. Hence, we get $a_{\ell;0} \neq 0$ and

$$d_\ell \;=\; a_{\ell;0} \;=\; \varphi_\ell(0).$$

Therefore, similar as above, in view of Theorem 3.1 and Lemma 3.1 it follows

$$d_{\ell-1} \;=\; d_\ell \sqrt{1+\frac{2}{\ell}}, \quad \ell = 1,\dots,n-1,$$

and particularly $d_{\ell-1} > d_\ell$. Hence, (b) is proven.

By (b) and $p = \varphi_n$, we have (note $\widetilde{p}^{[n]}(0) > p(0)$)

$$\begin{aligned} d_n\sqrt{1+\frac{2\varphi_n(0)}{nd_n}} \;&=\; \frac{1}{n}\big(\widetilde{p}^{[n]}(0)-p(0)\big)\sqrt{1+\frac{2p(0)}{\widetilde{p}^{[n]}(0)-p(0)}} \\ &=\; \frac{1}{n}\big(\widetilde{p}^{[n]}(0)-p(0)\big)\sqrt{\frac{\widetilde{p}^{[n]}(0)+p(0)}{\widetilde{p}^{[n]}(0)-p(0)}} \;=\; \frac{1}{n}\sqrt{\big(\widetilde{p}^{[n]}(0)\big)^2-\big(p(0)\big)^2}. \end{aligned}$$

Furthermore, by the principle of induction, one can show that

$$\prod_{j=\ell}^{n-1}\Big(1+\frac{2}{j}\Big) \;=\; \frac{n+n^2}{\ell+\ell^2}, \quad \ell = n-1,\dots,1.$$

Thus, using the coefficients as in (3), from (b) we get

$$\begin{aligned} a_{k;0} \;=\; \varphi_k(0) \;=\; d_k \;&=\; \sqrt{1+\frac{2\varphi_n(0)}{nd_n}}\prod_{\ell=k+1}^{n} d_\ell \\ &=\; \sqrt{\frac{1+n}{k+1+(k+1)^2}\Big(\big(\widetilde{p}^{[n]}(0)\big)^2-\big(p(0)\big)^2\Big)} \\ &=\; \sqrt{\frac{n+1}{k^2+3k+2}\Big(\big(\widetilde{p}^{[n]}(0)\big)^2-\big(p(0)\big)^2\Big)} \end{aligned}$$

for $k = 0,\dots,n-1$. Recalling $\varphi_\ell \in \mathcal{P}_{\ell;\mathsf{ar}}$ and $d_\ell = \varphi_\ell(0)$ for $\ell = 1,\dots,n-1$, the assertion of (c) follows.

Taking $p = \varphi_n$ and (7) into account the assertions of (d) and (e) are a consequence of (b). □

Corollary 3.2. *Suppose that $(\varphi_k)_{k=0}^{\tau}$ is a sequence of Szegő polynomials.*

(a) *If $\varphi_\ell \in \mathcal{P}_{\ell;\mathsf{ar}}$ for some index ℓ with $\tau > \ell \geq 1$, where the difference of consecutive coefficients of the polynomial φ_ℓ is not equal to $\varphi_\ell(0)$, then $\varphi_k \notin \mathcal{P}_{k;\mathsf{ar}}$ for all indices k with $\tau \geq k > \ell$.*

(b) *If $\varphi_\ell \in \mathcal{P}_{\ell;\mathsf{ar}}$ for some index ℓ with $\tau > \ell \geq 1$, where the associated Szegő parameter e_ℓ is zero, then $\varphi_k \notin \mathcal{P}_{k;\mathsf{ar}}$ for all indices k with $\tau \geq k > \ell$.*

Proof. Let $\varphi_\ell \in \mathcal{P}_{\ell;\mathsf{ar}}$ for an index ℓ with $\tau > \ell \geq 1$, where the difference d_ℓ of consecutive coefficients of φ_ℓ is different from $\varphi_\ell(0)$. Furthermore, we assume temporarily that there is an index k with $\tau \geq k > \ell$, where $\varphi_k \in \mathcal{P}_{k;\mathsf{ar}}$. Thus, Theorem 3.1 and part (b) of Proposition 3.1 with $p = \varphi_k$ yield $d_\ell = \varphi_\ell(0)$. But this conflicts with the condition of φ_ℓ. Therefore, $\varphi_k \notin \mathcal{P}_{k;\mathsf{ar}}$ for all indices k with $\tau \geq k > \ell$, and (a) is proven. Since the condition $e_\ell = 0$ leads to $\varphi_\ell(0) = 0$ (cf. Remark 2.2), the statement of (b) is a consequence of (a). □

In view of the interdependency of sequences of Szegő polynomials and orthonormal polynomial systems for measures $\mu \in \mathcal{M}$ (see, e.g., the notes from Remark 2.3 to the end of Section 2), we can simply reformulate the statements above in terms of orthonormal systems. In particular, we get the following result which turns out to be somewhat more surprising (if you do not have the recurrence relation in mind).

Theorem 3.2. *Suppose that $(\varphi_k)_{k=0}^{\tau}$ is an orthonormal polynomial system for some measure $\mu \in \mathcal{M}$.*

(a) *If $\varphi_n \in \mathcal{P}_{n;\mathsf{ar}}$ for some index n with $n \geq 2$, then $a_{k;0} \neq 0$ and*

$$a_{k;j} = \frac{u_k}{j+1}\sqrt{\frac{n+1}{k^2+3k+2}\big(a_{n;n}^2 - a_{n;0}^2\big)}, \quad j = 0, 1, \ldots, k,$$

with some $u_k \in \mathbb{T}$ for all indices $k = 0, 1, \ldots, n-1$ based on (3).

(b) *If $\varphi_\ell \in \mathcal{P}_{\ell;\mathsf{ar}}$ for some index ℓ with $\tau > \ell \geq 1$, where the difference of consecutive coefficients of the polynomial φ_ℓ is not equal to $\varphi_\ell(0)$, then $\varphi_k \notin \mathcal{P}_{k;\mathsf{ar}}$ for all indices k with $\tau \geq k > \ell$.*

(c) *If $\varphi_\ell \in \mathcal{P}_{\ell;\mathsf{ar}}$ for some index ℓ with $\tau > \ell \geq 1$, where $\varphi_\ell(0)$ is zero, then $\varphi_k \notin \mathcal{P}_{k;\mathsf{ar}}$ for all indices k with $\tau \geq k > \ell$.*

Based on Theorem 3.1, we can also see that the case of an infinite sequence of Szegő polynomials, where the coefficients of all polynomials are related to non-negative real arithmetic sequences, is a very special one.

Theorem 3.3. *Suppose that $(\varphi_k)_{k=0}^{\infty}$ is a sequence of Szegő polynomials and let $(e_\ell)_{\ell=1}^{\infty}$ be the associated sequence of Szegő parameters. Then the following statements are equivalent:*

(i) *There is a positive real number p_0 so that the sequence $(\varphi_k)_{k=0}^{\infty}$ is given by*

$$\varphi_k(x) = \frac{p_0}{\sqrt{k^2+3k+2}}\sum_{j=0}^{k}(j+1)x^j, \quad k = 0, 1, 2, 3, \ldots.$$

(ii) *For each index ℓ with $\ell \geq 1$ the polynomial φ_ℓ belongs to $\mathcal{P}_{\ell;\mathsf{ar}}$ and $\varphi_0(0)$ is a positive real number.*

(iii) *There is some $\ell_0 \geq 1$ so that for each index ℓ with $\ell \geq \ell_0$ the polynomial φ_ℓ belongs to $\mathcal{P}_{\ell;\mathsf{ar}}$.*

(iv) *The sequence* $(e_\ell)_{\ell=1}^{\infty}$ *is given by*

$$e_\ell \;=\; \frac{1}{\ell+1}, \qquad \ell = 1,2,3,\dots\,,$$

and $\varphi_0(0)$ *is a positive real number.*

In particular, if (i) *is fulfilled, then* $(\varphi_k)_{k=0}^{\infty}$ *is the normalized orthonormal polynomial system for the (uniquely determined) measure* μ *given by*

$$\mu(B) := \frac{1}{p_0^2\pi}\int\limits_B (1-\mathfrak{Re}\,z)\;\lambda(\mathrm{d}z), \qquad B\in\mathfrak{B}_{\mathbb{T}}, \tag{10}$$

where λ *stands for the linear Lebesgue-Borel measure on* $\mathbb{T}$.

Proof. The implications "(i) $\Rightarrow$ (ii)" and "(ii) $\Rightarrow$ (iii)" are a consequence of the settings. Furthermore, the implications "(ii) $\Rightarrow$ (iv)" and "(iii) $\Rightarrow$ (ii)" follow from Theorem 3.1. It remains to prove that "(iv) $\Rightarrow$ (i)". Therefore, we suppose that (iv) holds. Since $(\varphi_k)_{k=0}^{\infty}$ is a sequence of Szegő polynomials, for each positive integer n, the relation (1) is fulfilled with the special Szegő parameter e_n given by (iv). Taking into account that $\varphi_0(0)$ is a positive real number, i.e. φ_0 is the constant function with (positive) value $\varphi_0(0)$, we have

$$\varphi_0(x) \;=\; \varphi_0(0) \;=\; \frac{p_0}{\sqrt{0^2+3\cdot 0+2}}\sum_{j=0}^{0}(j+1)x^j$$

with the positive real number $p_0 := \sqrt{2}\varphi_0(0)$ and (1) for $n=1$ implies

$$\varphi_1(x) \;=\; \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}}\left(xp_0+\frac{1}{2}p_0\right) \;=\; \frac{p_0}{\sqrt{1^2+3\cdot 1+2}}\sum_{j=0}^{1}(j+1)x^j.$$

Now, by the principle of induction, we suppose that φ_k is given by

$$\varphi_k(x) \;=\; \frac{p_0}{\sqrt{k^2+3k+2}}\sum_{j=0}^{k}(j+1)x^j,$$

for some positive integer k. Then we have

$$\widetilde{\varphi}_k^{[k]}(x) \;=\; \frac{p_0}{\sqrt{k^2+3k+2}}\sum_{j=0}^{k}(j+1)x^{k-j},$$

where

$$\begin{aligned}\sqrt{1-\left(\frac{1}{k+2}\right)^2}\sqrt{k^2+3k+2} \;&=\; \frac{1}{k+2}\sqrt{\left((k+2)^2-1\right)(k^2+3k+2)}\\ &=\; \frac{1}{k+2}\sqrt{(k+1)(k+3)(k+1)(k+2)}\\ &=\; (k+1)\sqrt{\frac{k+3}{k+2}},\end{aligned}$$

$$j+\frac{k-j+1}{k+2} \;=\; \frac{jk+2j+k-j+1}{k+2} \;=\; \frac{(k+1)(j+1)}{k+2},$$

and

$$(k+2)(k+3) \;=\; k^2+5k+6 \;=\; (k+1)^2+3(k+1)+2.$$

Thus, in view of (iv) and (1) for $n=k+1$, we get

$$\begin{aligned}\varphi_{k+1}(x) \;&=\; \frac{1}{\sqrt{1-\left(\frac{1}{k+2}\right)^2}}\cdot\frac{p_0}{\sqrt{k^2+3k+2}}\cdot\Big(x\sum_{j=0}^{k}(j+1)x^j + \frac{1}{k+2}\sum_{j=0}^{k}(j+1)x^{k-j}\Big)\\ &=\; \frac{p_0}{k+1}\cdot\sqrt{\frac{k+2}{k+3}}\cdot\Big(\frac{k+1}{k+2}x^0+\sum_{j=1}^{k}\Big(j+\frac{k-j+1}{k+2}\Big)x^j + (k+1)x^{k+1}\Big)\\ &=\; \frac{p_0}{\sqrt{(k+2)(k+3)}}\sum_{j=0}^{k+1}(j+1)x^j \;=\; \frac{p_0}{\sqrt{(k+1)^2+3(k+1)+2}}\sum_{j=0}^{k+1}(j+1)x^j,\end{aligned}$$

so that we have proven, by the principle of induction, that (i) follows from (iv).

Suppose that (i) holds. If $p_0=\sqrt{2}$, then the considerations in [6, Example 1.6.4] imply that $(\varphi_k)_{k=0}^{\infty}$ is the normalized orthonormal polynomial system for the (uniquely determined) measure μ given by (10). Using this special case in combination with Remark 2.4, we see that this holds for any positive real number p_0. □

Finally, in addition to Lemma 3.1 and Corollary 3.2, we present the following result concerning the one-step extension given by (1).

Proposition 3.2. *Suppose that $\varphi_{n-1}\in\mathcal{P}_{n-1;\mathsf{ar}}$ for some $n\geq 2$ and let d_{n-1} be the difference of consecutive coefficients of the polynomial φ_{n-1}. Furthermore, let φ_n be the polynomial given by* (1) *and some parameter $e_n\in\mathbb{D}$, where*

$$\varphi_n(x) \;=\; a_{n;n}x^n + a_{n;n-1}x^{n-1} + \cdots + a_{n;1}x^1 + a_{n;0}$$

with some coefficients $a_{n;0},a_{n;1},\ldots,a_{n;n}\in\mathbb{C}$ as in (3).

(a) *The following statements are equivalent:*

   (i) *The parameter e_n is a real number with $0\leq e_n<1$ and $\varphi_{n-1}(0)=d_{n-1}$.*

   (ii) *The polynomial φ_n belongs to $\mathcal{P}_{n;\mathsf{ar}}$.*

   In particular, if (i) *is satisfied, then the coefficients $a_{n;0},a_{n;1},\ldots,a_{n;n}$ of the polynomial φ_n are real and*

$$a_{n;n} \;>\; a_{n;n-1} \;>\; \cdots \;>\; a_{n;1} \;>\; a_{n;0} \;\geq\; 0. \tag{11}$$

(b) *Suppose that $\varphi_{n-1}(0)\neq 0$, but also that $\varphi_{n-1}(0)\neq d_{n-1}$, and let*

$$m := \min\Big\{\frac{\varphi_{n-1}(0)}{d_{n-1}},\frac{d_{n-1}}{\varphi_{n-1}(0)}\Big\}.$$

   Then the following statements are equivalent:

(iii) *The parameter* e_n *is a real number with* $0 \leq e_n \leq m$.

(iv) *The coefficients* $a_{n;0}, a_{n;1}, \ldots, a_{n;n}$ *of the polynomial* φ_n *are real and*

$$a_{n;n} \geq a_{n;n-1} \geq \cdots \geq a_{n;1} \geq a_{n;0} \geq 0. \tag{12}$$

In particular, if (iii) *is satisfied, then* $\varphi_n \notin \mathcal{P}_{n;\mathsf{ar}}$*, although all zeros of* φ_n *belong to* $\mathbb{D}$ *and* $0 \leq e_n < m$ *is actually equivalent to* (11).

(c) *Suppose that* $\varphi_{n-1}(0) = 0$. *Then* $e_n = 0$ *holds if and only if the coefficients* $a_{n;0}, a_{n;1}, \ldots, a_{n;n}$ *of the polynomial* φ_n *are real and* (12) *holds. In particular, if* $e_n = 0$ *is satisfied, then* $a_{n;1} = a_{n;0} = 0$ *and all zeros of* φ_n *belong to* $\mathbb{D}$.

Proof. In view of Lemma 3.1 we see that (ii) implies (i). Now, we suppose that (i) holds. By $\varphi_{n-1} \in \mathcal{P}_{n-1;\mathsf{ar}}$ and $\varphi_{n-1}(0) = d_{n-1}$, we have the representation

$$\varphi_{n-1}(x) = \sum_{j=0}^{n-1} (j+1) d_{n-1} x^j.$$

Hence (cf. Remark 2.1), it follows that

$$\varphi_n(x) = \sum_{j=0}^{n} \frac{j d_{n-1} + e_n (n-j) d_{n-1}}{\sqrt{1-e_n^2}} x^j,$$

where

$$(j+1) d_{n-1} + e_n (n-j-1) d_{n-1} - \big(j d_{n-1} + e_n (n-j) d_{n-1}\big) = (1-e_n) d_{n-1}$$

for $j = 0, 1, \ldots, n-1$, i.e. we get (ii). Therefore, (i) and (ii) are equivalent. In particular, if (i) is satisfied, then (ii) as well so that the coefficients $a_{n;0}, a_{n;1}, \ldots, a_{n;n}$ of the polynomial φ_n are real and (11) holds. Thus, (a) is proven.

We now suppose that $\varphi_{n-1}(0) \neq 0$ and $\varphi_{n-1}(0) \neq d_{n-1}$ hold. In particular, due to (a), we have $\varphi_n \notin \mathcal{P}_{n;\mathsf{ar}}$. Recalling the choice of the number m according to (b) and $\varphi_{n-1} \in \mathcal{P}_{n-1;\mathsf{ar}}$, the equivalence of (iii) and (iv) is a consequence of [2, Proposition 5.4 (a)]. Furthermore, from [2, Proposition 5.4 (e)] and $\varphi_{n-1}(0) \neq 0$ we see that $0 \leq e_n < m$ is actually equivalent to (11). Taking $\varphi_{n-1} \in \mathcal{P}_{n-1;\mathsf{ar}}$ and Corollary 3.1 into account, by [2, Proposition 2.5 (a)] and the choice of φ_n it follows that all zeros of φ_n belong to $\mathbb{D}$. Thus, (b) is proven.

Finally, we suppose that $\varphi_{n-1}(0) = 0$ holds. Note that $e_n = 0$ and (1) imply

$$\varphi_n(x) = x \varphi_{n-1}(x).$$

Hence, if $e_n = 0$, then the coefficients $a_{n;0}, a_{n;1}, \ldots, a_{n;n}$ of the polynomial φ_n are real and (12) holds, where $a_{n;1} = a_{n;0} = 0$ (since $\varphi_{n-1} \in \mathcal{P}_{n-1;\mathsf{ar}}$ and $\varphi_{n-1}(0) = 0$). Furthermore, from Corollary 3.1 we know that all zeros of φ_{n-1}

belong to $\mathbb{D}$. Therefore, all zeros of φ_n belong to $\mathbb{D}$ when $e_n = 0$. Using [2, Proposition 5.4 (a)] one can also see that, if the coefficients $a_{n;0}, a_{n;1}, \dots, a_{n;n}$ of the polynomial φ_n are real and (12) holds, then $e_n = 0$. $\square$

As an aside to (11) and (12), we note that $a_{n;0} = 0$ is equivalent to $e_n = 0$ (cf. Remark 2.2). Furthermore, if $\varphi_{n-1}(0) = 0$, then (c) of Proposition 3.2 shows that there is only the choice $e_n = 0$ which realize (12) and no choice which realize (11).

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